The hydrodynamic stability of a thin film of liquid in uniform shearing motion

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The stability problem for a thin film of liquid having a linear mean-velocity profile and bounded by a fixed wall and free surface is solved asymptotically for large values of the Reynolds number R. The analysis is similar to that for plane Couette flow, but instability occurs for sufficiently large values of R in accordance with Heisenberg's criterion that neutral disturbances having finite wave numbers and phase velocities for $R = \infty$ are necessarily unstable as $R \rightarrow \infty$. It is found that a sufficient condition for stability is $W < 3$, where W is the Weber number based on the mean speed at the free surface and the depth of the film. The minimum critical Reynolds number, also based on free surface speed and film depth, is found to be $R = 203$. This last figure is in order-of-magnitude agreement with observation, but there remains considerable uncertainty as to whether the observed instability corresponds to that considered here. Neutral stability curves are presented in an R *vs* α ($=$ wave-number) plane with W as the family parameter. Brief consideration also is given to the time-rate-of-growth of unstable disturbances and to the lighter fluid that, in actual configurations, is responsible for the shear in the film. An appendix gives extended and more accurate results for the function $\mathscr{F}(z)$, introduced and calculated previously by Tietjens (1925) and Lin (1955).

1. Introduction

We present here an approximate determination of the conditions governing surface wave formation on a thin film of liquid that is bounded below by a wall and subjected to a prescribed shearing stress at its upper and otherwise free surface. This configuration approximates that arising in, for instance, film cooling and nose-cone ablation if we assume that the only significant role of the much lighter fluid flowing over the film is to produce the mean shear flow. Neglecting the effect of the lighter fluid on surface-wave formation obviously can be justified if its density is sufficiently small, but we must bear in mind that *sufficiently small* also implies that the kinematic viscosity of the film must be small compared with that of the lighter fluid (see § 8 below).

We also shall assume that the flow is incompressible, two-dimensional and laminar (so that the mean velocity profile is linear) and that only those waves for which the wave speed is equal to the mean flow speed at some point between the wall and the free surface can absorb energy from the mean flow.† Letting U_1 denote the velocity of the mean flow at the free surface, δ the mean thickness of the film, ν the kinematic viscosity, ρ the density, σ the surface tension, g the acceleration of gravity, and λ the wavelength of the surface wave, the dimensionless parameters characterizing the problem are:

$$R = U_1\delta/\nu \quad \text{(Reynolds number)}, \tag{1.1}$$

$$W = T^{-1} = \rho U_1^2\delta/\sigma \quad \text{(Weber number)}, \tag{1.2}$$

$$F = G^{-1} = U_1^2/g\delta \quad \text{(Froude number)} \tag{1.3}$$

and

$$\alpha = 2\pi\delta/\lambda \quad \text{(wave-number)}. \tag{1.4}$$

We shall assume $R \gg 1$ and $G \ll 1$ and seek the corresponding approximations to the neutral-stability (or simply *neutral*) curves for R *vs* α with T as the family parameter, say $R = R_c(\alpha; T)$.

The foregoing problem has been studied by Feldman (1957), who included the lighter fluid in his stability analysis on the assumption that its mean velocity profile could be regarded as linear and semi-infinite in extent (a reasonable approximation if $\alpha\delta_1 \gg 1$, where δ_1 denotes the thickness of the laminar sublayer in this fluid). Feldman's configuration is characterized by R, W, F, α and, in addition, r, the ratio of density of the lighter fluid to that of the lower fluid, and $1/m$, the corresponding viscosity ratio. This configuration presumably should reduce to that of the preceding paragraph as $r \to 0$ with m fixed. Feldman found that there was stability for all α in this limit, that both gravity and surface tension were destabilizing for $r \ll 1$, and that the minimum value of R_c for air blowing over an 0·005 in. film of water was an order of magnitude larger than that inferred from observation.

A problem closely related to that of the penultimate paragraph is presented by a laminar film on an inclined plane, the stability of which has been studied by Benjamin (1957). The essential differences are that the velocity profile for the latter problem is parabolic rather than linear and that the Reynolds and Froude numbers are not independent parameters (for a fixed angle of inclination). Benjamin gave special consideration to a vertical film and found instability for all R and sufficiently small α. He also found that surface tension was stabilizing and calculated a wavelength for maximum instability that was in close agreement with observation.

It might be thought that the very striking disparity between the respective results of Feldman and Benjamin is analogous to the disparity between plane Couette and plane Poiseuille flow and could be traced to the important role of profile curvature. In fact, the analogy between the film with a free surface and plane Couette flow is not appropriate, for it overlooks Heisenberg's criterion (as formulated by Lin 1946) that 'if a velocity profile has an "inviscid" neutral disturbance with non-vanishing wave-number and phase velocity (equal to the velocity of the mean flow at some point in the profile), the disturbance with the same wave-number is unstable in the real fluid when the Reynolds number is sufficiently large'. Lin's proof of Heisenberg's criterion is not directly applicable to the present configuration, but it is readily extended. In Lin's words (L 4.5,

† See §8 below for an *a posteriori* justification of this last assumption.

where the prefix L, followed by the appropriate section or equation number, refers to Lin's 1955 monograph), 'The key mechanism is a shift in the phase of the two components of the velocity of the oscillation by the viscous forces at the solid boundary. This produces a Reynolds stress which converts energy from the basic flow to the disturbance'. This energy is $O(R^{-\frac{1}{2}})$ as $R \to \infty$, whereas the energy dissipated in the neighbourhood of the free surface is $O(R^{-1})$, while that dissipated elsewhere in the flow is $O(R^{-\frac{3}{2}})$. We also observe (see § 4 below) that the film under consideration can support inviscid surface waves with two permissible wave speeds for each wave-number, one upstream and the other downstream relative to U_1, and that the former must lie in $(0, U_1)$ relative to the wall if U_1 is sufficiently large. It follows that the flow must be unstable for sufficiently large values of R, W and F (W and F being the appropriate measures of U_1 for $R = \infty$). Conversely, if either W or F is sufficiently small (T or G sufficiently large) no wave speed exists in $(0, U_1)$ relative to the wall, and we therefore should expect sufficiently large surface tension and/or gravity to be stabilizing.

It appears likely that the primary reason for the conflict between the foregoing arguments and Feldman's conclusions lies in the inadequacy of his asymptotic (as $R \to \infty$) approximations for small wave speeds (cf. L 3.4 and 3.6). Even this would not account for his conclusion of stability for all α as $r \to 0$, however, so that either his analysis contained mathematical errors or he simply misinterpreted the significance of the result that, for *his* asymptotic approximation, $R_c \to \infty$ as $r \to 0$—namely, that disturbances having wave speeds in $(0, U_1)$ are unstable for sufficiently large, finite values of R.

The argument outlined in the penultimate paragraph requires three important qualifications if the presence of a *laminar* flow in the lighter fluid is recognized. First, there may be an energy dissipation of $O(rR_g^{-\frac{1}{2}}) = O(r^{\frac{1}{2}}m^{-\frac{1}{2}}R^{-\frac{1}{2}})$ in this fluid ($R_g = rmR$ being its Reynolds number). This is especially important for large α (short wavelength), where the energy supply from the wall actually is found to be $O(e^{-2\alpha}R^{-\frac{1}{2}})$ at the interface. Secondly, the film of denser fluid will act essentially as a wall with respect to the viscous forces in the lighter fluid, whence energy may be supplied to disturbances having wave speeds in excess of U_1 (cf. Benjamin 1959). Thirdly, and perhaps unexpectedly, the lighter fluid may significantly alter the wave speeds as $R \to \infty$ if its kinematic viscosity is of the same order as or *less* than that of the film. We shall carry (in § 8) the mathematical analysis of the two-fluid problem far enough to elucidate these three effects, but we shall not attempt to present numerical results for the neutral curves (except, of course, for $r = 0$). These doubtless would be exceedingly complex in consequence of the two, distinct classes of modes (finding their primary source of energy in one fluid or the other) and might be expected to resemble the neutral curves obtained by Lock (1954) for two semi-infinite fluids with *curved* velocity profiles.

It seems likely that, in the majority of practical applications, the flow in the lighter fluid would be turbulent and that the laminar sublayer would not be large compared with significant wavelengths (see § 8 below). Energy then could be transferred from the mean flow in this upper fluid to disturbances in the film both through the direct action of turbulent fluctuations (Phillips 1957), and through the Reynolds stress associated with profile curvature (Miles 1957, 1959, 1960). Limited experimental data (Knuth 1954) indicate that disturbances of relatively short wavelength are present for all film speeds, while stronger disturbances of relatively long (compared with film thickness) wavelength appear only when a rather well-defined, critical Reynolds number is exceeded. It is at least plausible to associate the former disturbances with the direct action of turbulent fluctuations and the latter with hydrodynamic instability. Whether this hydrodynamic instability depends primarily on energy transfer from the mean flow in the film (as described herein) or that in the upper fluid depends decisively on whether the phase velocity of the disturbance is less or greater than the interfacial speed.

The minimum critical Reynolds number (approximately 200) determined in the following analysis is in order-of-magnitude agreement with that observed by Knuth (1954), but the corresponding wavelengths are much smaller than those observed by him (private communication). Knuth's observations of phase velocity are inconclusive, but it appears that at least some of the larger, unstable waves were travelling downstream relative to the interface and hence did not correspond to those considered here. Further experiments, with special emphasis on wave kinematics, evidently are desirable.

2. Equations of motion

Consider the flow configuration sketched in figure 1. We designate the horizontal and vertical components of velocity by u and v, the hydrodynamic pressure by p, and the shear stress by τ and refer all lengths, velocities and stresses to δ, U_1 and ρU_1^2. The equilibrium flow then is specified by

$$u = U(y) = y, \quad p = p_0, \quad \tau = \tau_0 = R^{-1}, \qquad (2.1\,a,b,c)$$

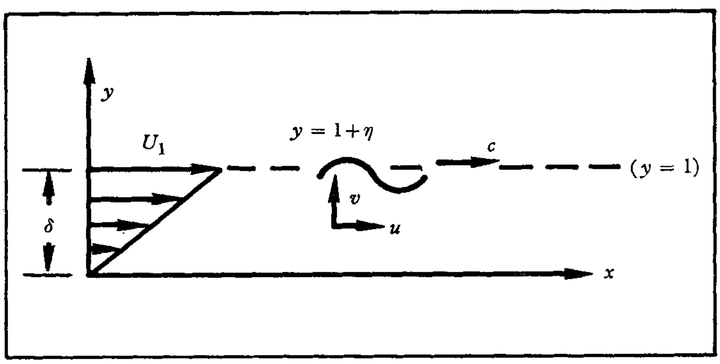


FIGURE 1. Sketch of the linear shear flow and surface-wave disturbance.

where p_0 denotes the static pressure. We superimpose on this equilibrium flow a small, travelling-wave disturbance having the (x, t)-dependence

$$E(x,t) = e^{i\alpha(x-ct)}, \qquad (2.2)$$

where α denotes the wave-number (positive by definition) of (1.4) and c the wave speed, and we seek to determine those values of c, namely the eigenvalues, that are compatible with the equations of motion and the boundary conditions.

Requiring c to be real and in the interval (0, 1) then yields the neutral curves, R_c *vs* α.

We may satisfy the continuity equation by introducing a perturbation stream function

$$\psi(x,y,t) = \phi(y)\,E(x,t) \tag{2.3}$$

such that the perturbation velocity is given by

$$u = -\psi_y = -\phi' E, \quad v = \psi_x = i\alpha\phi E, \tag{2.4 a, b}$$

where the prime implies differentiation with respect to y. We also remark that the displacement η of any streamline—in particular the free surface or interface at $y \doteqdot 1$—is given by

$$\psi = (U-c)\,\eta. \tag{2.5}$$

Posing (2.4 *a*, *b*) in the equations of motion (cf. L 1.3), we obtain

$$(U-c)(\phi''-\alpha^2\phi) = (i\alpha R)^{-1}(\phi^{\mathrm{iv}}-2\alpha^2\phi''+\alpha^4\phi), \tag{2.6}$$

$$p = [(U-c)\,\phi' - U'\phi - (i\alpha R)^{-1}(\phi'''-\alpha^2\phi')]\,E, \tag{2.7}$$

$$\tau = R^{-1}(u_y+v_x) = -R^{-1}(\phi''+\alpha^2\phi)\,E, \tag{2.8}$$

for the determination of ϕ and the perturbation pressure and shear stress.

The boundary conditions are

$$u = v = 0, \quad y = 0, \tag{2.9 a, b}$$

$$-p+2R^{-1}v_y = T\eta_{xx} - G\eta, \quad \tau = 0, \quad y = 1, \tag{2.9 c, d}$$

corresponding to the requirements that the velocity vanish at the wall, that the normal stress at the surface be in equilibrium with the capillary and gravitational forces resisting the displacement η, and that the shear stress vanish at the surface. Substituting (2.4 *a*, *b*), (2.5), (2.7) and (2.8) in (2.9 *a*, *b*, *c*, *d*), we obtain

$$\phi = \phi' = 0, \quad y = 0, \tag{2.10 a, b}$$

$$\varpi = (1-c)\,\phi' - \phi - (i\alpha R)^{-1}(\phi''' - 3\alpha^2\phi') - (T\alpha^2+G)(1-c)^{-1}\phi = 0, \quad y = 1, \tag{2.10 c}$$

$$\phi''+\alpha^2\phi = 0, \quad y = 1, \tag{2.10 d}$$

where we have introduced the abbreviation ϖ for convenience in the subsequent analysis.

3. Asymptotic solutions

We may determine four, linearly independent solutions to (2.6) with $U = y$ therein according to [see F (6)–(17) for details, where the prefix F denotes an equation in Feldman's paper]

$$\phi_1 = \cosh(\alpha y), \quad \phi_2 = \sinh(\alpha y), \tag{3.1 a, b}$$

$$\phi_{3,4} = \frac{1}{\alpha}\int_{\pm\infty}^{y} \sinh[\alpha(y-y')]\,f_{3,4}(\zeta')\,dy', \tag{3.2}$$

$$f_{3,4}(\zeta) = \zeta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1,2)}[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}], \tag{3.3}$$

$$\zeta = (\alpha R)^{\frac{1}{3}}(y-c) - i\alpha^2(\alpha R)^{-\frac{2}{3}}, \tag{3.4}$$

where the upper and lower signs in (3.2) correspond to ϕ_3 and ϕ_4, respectively, and ζ' is given by (3.4) with y replaced by y' therein. We designate $\phi_{1,2}$ as the *inviscid* solutions and $\phi_{3,4}$ as the *viscous* solutions.

We require asymptotic approximations to $\phi_{3,4}$ as $\alpha R \to \infty$. The simplest such approximations are [cf. L (3.4.9)]

$$\phi_{3,4} \sim \text{const.} \times (y-c)^{-\frac{5}{4}} \exp\left[\mp \tfrac{2}{3}(i\alpha R)^{\frac{1}{2}} (y-c)^{\frac{3}{2}}\right], \quad \alpha R\,|y-c| \to \infty, \tag{3.5}$$

where $i^{\frac{1}{2}} = \exp(i\pi/4)$ and [cf. L (3.4.11)]

$$y-c = (c-y)\,e^{-i\pi} \quad (y < c). \tag{3.6}$$

We shall assume $0 < c < 1$ and (subject to *a posteriori* verification)

$$(\alpha R)^{\frac{1}{2}} (1-c)^{\frac{3}{2}} \gg 1, \tag{3.7}$$

so that (3.5) governs the behaviour of both ϕ_3 and ϕ_4 in the neighbourhood of the free surface; on the other hand, we cannot use (3.5) in the neighbourhood of the wall unless $(\alpha R)^{\frac{1}{2}} c^{\frac{3}{2}} \gg 1$. Irrespective of whether or not this last inequality is satisfied, however, (3.7) is sufficient to justify the neglect of ϕ_4 in the boundary conditions at $y = 0$ and ϕ_3 in the boundary conditions at $y = 1$, the error being $O\{\exp[-\frac{1}{3}(2\alpha R)^{\frac{1}{2}} (1-c)^{\frac{3}{2}}]\}$ relative to unity.

An asymptotic approximation to ϕ_3 that is uniformly valid with respect to c as $\alpha R \to \infty$ is [cf. L (3.6.4)]

$$\phi_3 \sim (\alpha R)^{-\frac{2}{3}} \int_{\infty}^{(\alpha R)^{1/3}(y-c)} [(\alpha R)^{\frac{1}{3}} (y-c) - \zeta]\, f_3(\zeta)\, d\zeta. \tag{3.8}$$

We shall base the subsequent analysis on this approximation, rather than (3.5), and it is primarily (but not only) in this that our analysis differs from that of Feldman.

4. Eigenvalue equation

We may obtain the eigenvalue equation for c by substituting a linear superposition of $\phi_{1,2,3,4}$ in the boundary conditions $(2.10\,a, b, c, d)$ and equating the determinant of the resulting algebraic equations (in the unknown coefficients of $\phi_{1,2,3,4}$) to zero. Invoking the approximation that ϕ_3 and ϕ_4 are exponentially small at the surface and at the wall, respectively, the result is

$$\begin{vmatrix} \phi_{10} & \phi_{20} & \phi_{30} & 0 \\ \phi'_{10} & \phi'_{20} & \phi'_{30} & 0 \\ \varpi_1 & \varpi_2 & 0 & \varpi_4 \\ 2\alpha^2\phi_{11} & 2\alpha^2\phi_{21} & 0 & \phi''_{41} + \alpha^2\phi_{41} \end{vmatrix} = 0, \tag{4.1}$$

where the first subscript $(1, 2, 3, 4)$ identifies the individual solution and the second the point of evaluation (0 or 1).

It is immediately evident from (4.1) that the eigenvalue equation depends on the viscous solutions only through ϕ_{30}/ϕ'_{30} and $\varpi_4/(\phi''_{41} + \alpha^2\phi_{41})$. The former quantity is given by (3.8) as

$$\frac{\phi_{30}}{\phi'_{30}} = -(\alpha R)^{-\frac{1}{3}} zF(z), \tag{4.2}$$

where

$$z = (\alpha R)^{\frac{1}{3}} c \tag{4.3}$$

and

$$F(z) = 1 + \frac{\displaystyle\int_{\infty}^{-z} \zeta f_3(\zeta)\, d\zeta}{\displaystyle z\int_{\infty}^{-z} f_3(\zeta)\, d\zeta} \tag{4.4}$$

is defined as in L 3.6 (although differing from Lin's form to the extent of integration by parts). The latter quantity is given by (2.10c) and (3.5) as

$$\frac{\varpi_4}{\phi''_{41}+\alpha^2\phi_{41}} = -\frac{(T\alpha^2+G)}{i\alpha R(1-c)^2}+O(R^{-\frac{2}{3}}). \tag{4.5}$$

Substituting (3.1 a, b), (4.2) and (4.5) in (4.1) and expanding the determinant, we may place the result in the form

$$[\alpha \coth\alpha+\alpha^2 \operatorname{cosech}^2\alpha F(z)\, c]\,(1-c-2i\alpha R^{-1})^2 \\ -(1-c-2i\alpha R^{-1})-(T\alpha^2+G)+O(\alpha^2 \operatorname{cosech}^2 \alpha R^{-\frac{1}{3}}, R^{-\frac{2}{3}}) = 0. \tag{4.6}$$

We remark that the terms of $O(R^{-1})$ in (4.6) need be retained only if α is large; otherwise, they are of higher order than terms already neglected. If, on the other hand, α is large, the values of $c-1$ (the wave speed relative to the free surface) given by (4.6) reduce to those calculated by Stokes (1850; Lamb 1945, § 348) for surface waves on a slightly viscous, deep liquid. We also remark that neglecting the terms of $O(R^{-1})$ in (4.6) is tantamount to suppressing the boundary condition of zero shear at the free surface.

5. The inviscid problem

Letting $R = \infty$ in (4.6), we obtain

$$\alpha \coth\alpha(1-c)^2-(1-c)-(T\alpha^2+G) = 0 \tag{5.1}$$

as the eigenvalue equation governing the inviscid problem. The roots of (5.1), to which we find it convenient to assign the subscript zero, are

$$c_0 = 1-\left\{\frac{1\pm[1+4(T\alpha^2+G)\,\alpha\coth\alpha]^{\frac{1}{2}}}{2\alpha\coth\alpha}\right\}. \tag{5.2}$$

We note that the restoration of the original, physical parameters, in place of their dimensionless counterparts, yields

$$c_0-U_1 = \mp\left[\left(\frac{\sigma}{\rho}k+\frac{g}{k}\right)\tanh{(k\delta)}+\left(\frac{U_1\tanh k\delta}{2k\delta}\right)^2\right]^{\frac{1}{2}}-\left(\frac{U_1\tanh k\delta}{2k\delta}\right), \tag{5.3 a}$$

$$k = 2\pi/\lambda, \tag{5.3 b}$$

for the inviscid wave speeds relative to the free surface.

The eigenvalues given by (5.2) are real in consequence of the fact that there can be no energy transfer between an inviscid shear flow and a travelling wave disturbance in the absence of profile curvature (L 4.3). That eigenvalue corresponding to the negative radical lies in $c > 1$ and, by hypothesis, is not significant for our stability problem.† That root corresponding to the positive radical lies in (0, 1) if and only if

$$\alpha\coth\alpha-1-T\alpha^2-G > 0, \tag{5.4}$$

† Feldman discarded the root in $c > 1$ on the basis of the erroneous argument that it was algebraically extraneous. We also note that F (71) and F (72) are identical and not, as Feldman states, in disagreement. Cf. (8.7) below.

as will be true if $\alpha_0 < \alpha < \alpha_1$, where α_0 and α_1 are the roots of

$$f(\alpha) = T + G\alpha^{-2}, \tag{5.5}$$

and

$$f(\alpha) = \alpha^{-1}\coth\alpha - \alpha^{-2}. \tag{5.6}$$

The function $f(\alpha)$ is plotted in figure 3.

We can prove that (5.5) has either two or no real roots by noting that $f(\alpha)$ decreases monotonically from $\frac{1}{3}$ to 0 as α increases from 0 to α, while $T + G\alpha^{-2}$ decreases from ∞ to T. If $T < \frac{1}{3}$ and $G \ll 1$, α_0 and α_1 are determined by

$$\alpha_0^2 = (\tfrac{1}{3} - T)^{-1} G, \quad f(\alpha_1) = T. \tag{5.7a,b}$$

We infer that $T > \frac{1}{3}$ ($W < 3$) *is a sufficient condition for stability as* $R \to \infty$, since then no eigenvalue of c exists in (0, 1). We emphasize, however, that the presence of an upper fluid generally would alter this conclusion either by modifying the condition for $c < 1$ (see § 8 below) or by rendering unstable some modes for which $c > 1$.

6. The neutral curves

We may determine the neutral curves by assuming c to be real in (4.6) and equating the real and imaginary parts separately to zero to obtain

$$[\alpha\coth\alpha + \alpha^2\operatorname{cosech}^2\alpha\, F_r(z)\, c]\,(1-c)^2 - (1-c) - (T\alpha^2 + G) = 0 \tag{6.1}$$

and

$$F_i(z)\,\alpha^2\operatorname{cosech}^2\alpha\, c(1-c)^2 - 2\alpha R^{-1}\{2(1-c)\,[\alpha\coth\alpha + \alpha^2\operatorname{cosech}^2\alpha\, F_r(z)\, c] - 1\} = 0, \tag{6.2}$$

where the subscripts r and i denote real and imaginary parts. Substituting

$$R = z^3 c^{-3}\alpha^{-1} \tag{6.3}$$

and rearranging, we obtain

$$z^3 F_i(z) = 2\left(\frac{c}{1-c}\right)^2 \{2(1-c)\,[\alpha\coth\alpha + \alpha^2\operatorname{cosech}^2\alpha\, F_r(z)\, c] - 1\}\sinh^2\alpha. \tag{6.4}$$

Now, within the approximations already invoked, we may neglect the term in $F_r c$ and replace c by the inviscid approximation c_0 on the right-hand side of (6.4) to obtain

$$z^3 F_i(z) = H(\alpha, T + G\alpha^{-2}), \tag{6.5}$$

where

$$H(\alpha, T) = 2\left[\frac{2\alpha\cosh\alpha}{1 + (1 + 4T\alpha^3\coth\alpha)^{\frac{1}{2}}} - \sinh\alpha\right]^2 (1 + 4T\alpha^3\coth\alpha)^{\frac{1}{2}}. \tag{6.6}$$

We emphasize that the retention of $F_r(z)$ in (6.1) is significant, even though we have neglected it in (6.5) and (6.6).

If α is small, (6.1) yields

$$c = \left(\frac{\frac{1}{3} - T}{1 - F_r}\right)(\alpha^2 - \alpha_0^2) \quad (\alpha_0 < \alpha \ll 1), \tag{6.7}$$

where α_0 is given by (5.7*a*). It follows that the right-hand side of (6.4) is $O(\alpha^6)$, whence we may approximate (6.5) by $F_i = 0$, which implies $z = 2{\cdot}294$ and $F_r = 0{\cdot}5637$. We then obtain

$$R = 27{\cdot}16(1 - 3T)^{-3}(\alpha^2 - \alpha_0^2)^{-3}\alpha^{-1} \tag{6.8}$$

from (6.3) and (6.7). We note that if $\alpha_0 = 0$ ($G = 0$), $R \sim \alpha^{-7}$ as $\alpha \to 0$, and the shape of the lower branch of the neutral curve then is similar to that for plane Poiseuille flow [L (3.6.12)]. We observe, however, that our neutral curves are single-valued as $\alpha \to 0$, the *upper* branch of the Poiseuille curve in this limit being determined by profile curvature [L (3.6.13)].

If $\alpha \gg \alpha_0$, we either may ignore G or may carry out our calculations for fixed values of T in (6.6) with the understanding that T may be replaced by $T + G\alpha^{-2}$ to account for G. To obtain a point on a neutral curve for fixed T, we may proceed as follows: (i) select α; (ii) determine H from (6.6); (iii) determine z from (6.5), the solution of which may not be single-valued (see below); (iv) determine c from (6.1), with $G = 0$ therein and $0 < c < 1$; (v) determine R from (6.3).

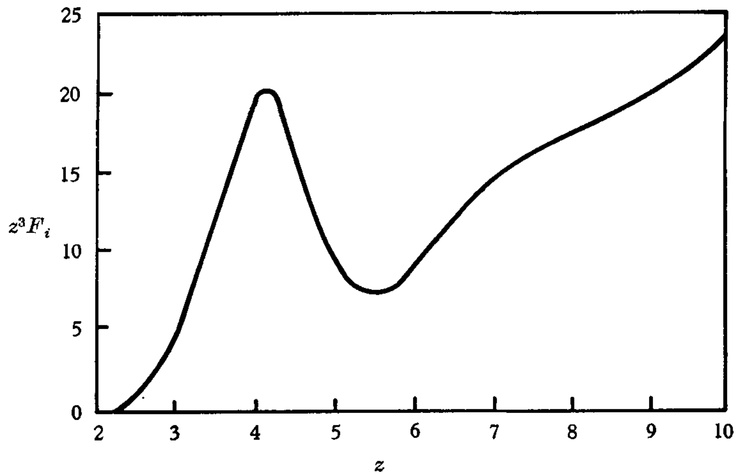


FIGURE 2. The function $z^3F_i(z)$, required for the solution of (6.5).

The left-hand side of (6.5), $z^3F_i(z)$, is plotted in figure 2 and is found to be positive for $z > 2{\cdot}294$, to rise to a maximum of 19·9 at $z = 4{\cdot}10$, to decrease to a minimum of 7·05 at $z = 5{\cdot}44$, and then to increase monotonically, being asymptotic to $2^{-\frac{1}{2}}z^{\frac{3}{2}}$.

The right-hand side of (6.5), $H(\alpha, T)$, vanishes at $\alpha = \alpha_0$ and $\alpha = \alpha_1$ and has a single maximum in (α_0, α_1) unless $T = 0$ ($\alpha_1 = \infty$) in which case it increases monotonically with α. It follows that the neutral curve will have either one or three branches and that we may determine the (α, T) regions in which these possibilities occur by plotting,

$$H(\alpha, T) = (z^3F_i)_{\min,\max}, \tag{6.9}$$

as shown in figure 3. If $0{\cdot}172 < T < 0{\cdot}184$ (approximate) a portion of the neutral curve will take the form of a closed loop, the interior of which forms an island of stability in an (α, R)-plane. If $T > 0{\cdot}184$ the neutral curves are single-valued.

Neutral curves were determined according to the foregoing procedure on a high-speed digital computer for $T = 0\,(0{\cdot}05)\,0{\cdot}15\,(0{\cdot}01)\,0{\cdot}20\,(0{\cdot}05)\,0{\cdot}30$. Neutral curves for $T = 0$, 0·10, 0·20, 0·25 and 0·30 are plotted in figure $4a$, and that for

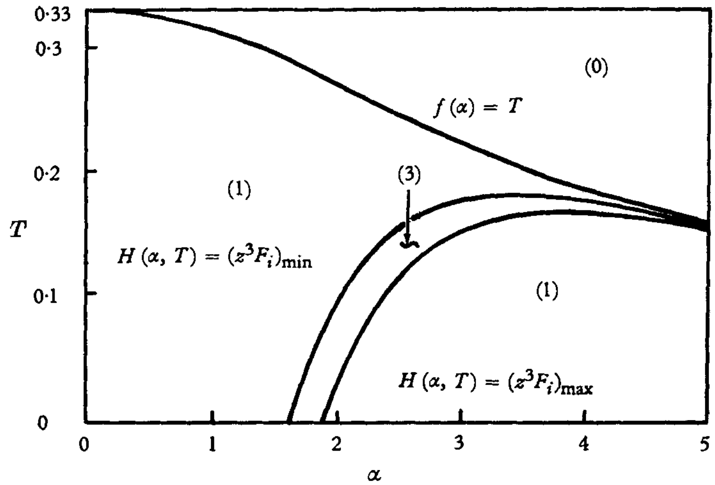


FIGURE 3. The number of branches of the neutral curves in the various α, T-regions is indicated in parentheses.

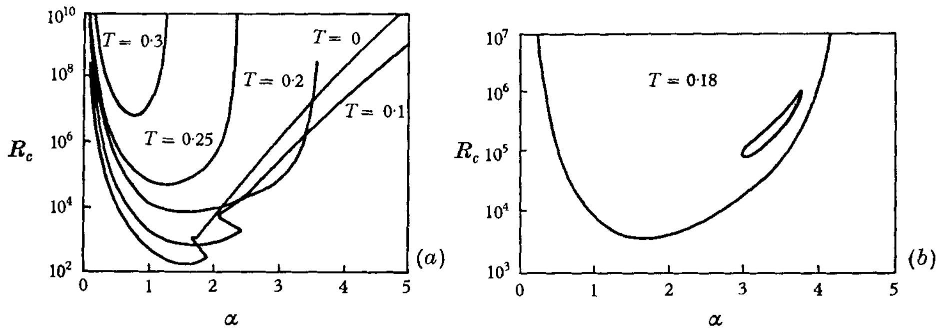


FIGURE 4a. The neutral stability curves determined by (6.1) and (6.5) for $G = 0$. If $G > 0$ the parameter labelled T may be replaced by $T + G\alpha^{-2}$.

FIGURE 4b. A neutral stability curve with an island of stability.

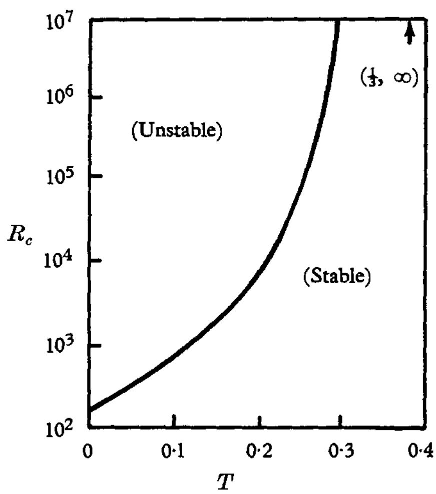


FIGURE 5. The (T, R)-locus of stability, as determined from the minima of the neutral curves.

$T = 0{\cdot}18$, illustrating an island of stability, in figure 4*b*.† The minimum values of the neutral curves are plotted versus T in figure 5. We remark that, although increasing T has a stabilizing effect for all α below α_c (where α_c indicates the location of R_c) and for sufficiently large α, there exist intermediate ranges of α for which increasing T is destabilizing.

7. Determination of most unstable mode

If $R > R_c$ the disturbance of (2.2) will grow like $\exp(\alpha c_i t)$, and the most unstable mode is that for which αc_i is a maximum.

We may obtain an approximation to αc_i that is at least qualitatively accurate by assuming $|c| \ll 1$ in (4.6) and letting $G = 0$, whence

$$c = \frac{\alpha \coth \alpha - 1 - T\alpha^2}{2\alpha \coth \alpha - 1 - \alpha^2 \operatorname{cosech}^2 \alpha \, F(z)} - \frac{2i\alpha}{R} \tag{7.1 a}$$

$$= \frac{[f(\alpha) - T]\mathscr{F}(z) \sinh^2 \alpha}{1 + [1 - \alpha^2 f^2(\alpha)]\mathscr{F}(z) \sinh^2 \alpha} - \frac{2i\alpha}{R}, \tag{7.1 b}$$

where $f(\alpha)$ is defined by (4.4) and

$$\mathscr{F}(z) = [1 - F(z)]^{-1}. \tag{7.2}$$

Assuming $|c_i| \ll c_r$, approximating $\mathscr{F}$ according to

$$\mathscr{F}(z_r + iz_i) = \mathscr{F}_r(z_r) + i[\mathscr{F}_i(z_r) + z_i \mathscr{F}_r'(z_r)] + O(z_i^2 \mathscr{F}_r'', z_i \mathscr{F}_i), \tag{7.3}$$

separating the real and imaginary parts of (7.1 *b*), assuming $\alpha \gg 1$ in the resulting coefficient of R^{-1} (since it is only for $\alpha \gg 1$ that this term is significant), and eliminating c_r through

$$z_r = (\alpha R)^{\frac{1}{3}} c_r, \tag{7.4}$$

we obtain

$$z_r = \frac{R^{\frac{1}{3}} \alpha^{\frac{1}{3}} [f(\alpha) - T]\mathscr{F}_r \sinh^2 \alpha}{1 + [1 - \alpha^2 f^2(\alpha)]\mathscr{F}_r \sinh^2 \alpha} \tag{7.5 a}$$

and

$$\alpha c_i = \frac{R^{-\frac{1}{3}} (z_r \mathscr{F}_i / \mathscr{F}_r) \alpha^{\frac{2}{3}}}{1 + [1 - \alpha^2 f^2(\alpha)]\mathscr{F}_r \sinh^2 \alpha - (z_r \mathscr{F}_r' / \mathscr{F}_r)} - 2R^{-1}\alpha^2, \tag{7.5 b}$$

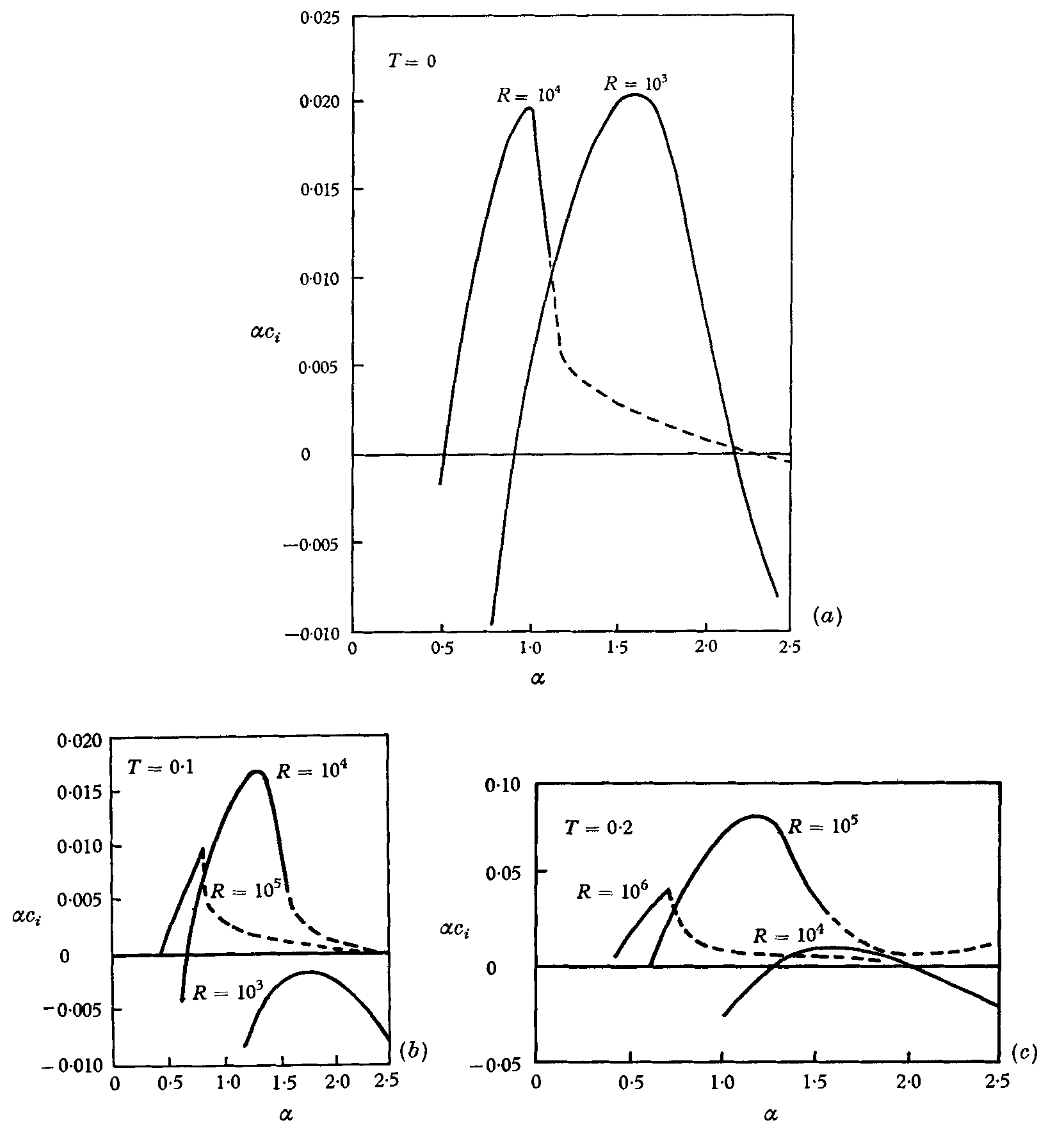
where the argument of $\mathscr{F}_r$, $\mathscr{F}_i$, and $\mathscr{F}_r'$ is z_r. We remark that, since (see Appendix)

$$\mathscr{F}(z) \sim 1 + i(2\alpha R)^{-\frac{1}{2}} c_r^{-\frac{3}{2}}, \tag{7.6}$$

$c_i > 0$ (provided that $0 < c_r < 1$) as $R \to \infty$, in accordance with Heisenberg's criterion.

Some numerical results computed from (7.5 *a*, *b*) are plotted in figure 6. The dashed portions of the curve are based on the asymptotic approximation (7.6) for $z > 5$ and are less accurate (see Appendix) than the solid portions, which were based on Lin's tabulated values for $\mathscr{F}(z)$ (and were computed prior to the recalculation of $\mathscr{F}$ described in the appendix). It appears that the maxima of the curves, where they are well defined, occur at values of α between 1 and $1{\cdot}5$, corresponding to wavelengths between 6 and 4 film thicknesses.

† The complete numerical results are tabulated in Table 2 of Report STL-TR-59-000-00892, Space Technology Laboratories, Inc., Los Angeles, California.



FIGURES 6*a*, *b*, *c*. The time-rate-of-growth of disturbances, as calculated from (7.5). The dashed portions of the curves are based on the asymptotic approximation (7.6).

8. Effect of upper fluid

We now consider quite briefly the dynamical effects of the upper fluid on the assumption that it too has a linear velocity profile. This consideration affords some clarification of the significance of the preceding analysis for small but finite values of the density ratio

$$r = \rho_g/\rho_l \tag{8 1}$$

and of the importance of the viscosity ratios

$$m = \mu_l/\mu_g, \quad n = rm = \nu_l/\nu_g. \tag{8.2a, b}$$

We emphasize that the assumption of a linear velocity profile in the upper fluid is likely to be valid only for very short wavelengths. Assuming aerodynamically smooth flow in the upper fluid, the thickness of the laminar sublayer is given approximately by

$$\delta_g = 5\nu_g/U_*, \tag{8.3}$$

where $\rho_g U_*^2$ is the shearing stress at the interface. Equating this stress to $\mu_l U_1/\delta$ and eliminating U_*, we obtain

$$\alpha_g = 2\pi\delta_g/\lambda = 5(rR)^{-\frac{1}{2}} m^{-1}\alpha. \tag{8.4}$$

Anticipating that the dynamical effects of the flow outside of the laminar sublayer would be $O[\exp(-2\alpha_g)]$, we conclude that $\alpha_g > 1$ might justify the assumption of a linear velocity profile in the upper fluid. This condition certainly was not satisfied in Knuth's experiments, and it seems unlikely that it would be satisfied in most configurations of practical interest.

Repeating Feldman's analysis except as regards the asymptotic approximation of the viscous solutions, and assuming $m \gg 1$, we obtain the asymptotic (as $R \to \infty$) eigenvalue equation

$$[(4.6)] + [n(1-c) - r(1-c)^2 \alpha\, e^{\alpha} \operatorname{cosech} \alpha\, F(Z)]\,[1 + O(m^{-1})] = 0, \tag{8.5}$$

where $[(4.6)]$ stands for the left-hand side of equation (4.6), F is given by (4.4), and

$$Z = -\left(\frac{\alpha R r}{m}\right)^{\frac{1}{3}} (1-c) = -\left(\frac{r}{m}\right)^{\frac{1}{3}} \left(\frac{1-c}{c}\right) z. \tag{8.6 a, b}$$

Let us consider first the values of c for $R = \infty$, where (8.5) yields

$$\alpha \coth \alpha (1-c)^2 - (1-n)(1-c) - (T\alpha^2 + G) = 0 \tag{8.7}$$

in place of (5.1). If $n \ll 1$ its only significant effect is to reduce α_0^2, as may be seen by replacing G by $G-n$ in (5.7 a). If $n < 1$ but is not small it also raises the value of α_1 (the upper limit of α for which $c > 0$ is possible), but otherwise it does not affect the disposition of the roots relative to 0 and 1. If $n > 1$ the disposition of the roots is affected; for example, if $T\alpha^2 + G \ll n-1$ the roots to (8.7) are given approximately by

$$1 - c \doteqdot \frac{T\alpha^2 + G}{n-1}, \quad -(n-1)\frac{\tanh \alpha}{\alpha}. \tag{8.8 a, b}$$

Small, positive values of c then are impossible, and the primary instability may be associated with a mode for which $c > 1$, in which case the upper fluid no longer can be relegated to a subsidiary role, even though $r \ll 1$; moreover, the restriction (3.7) may break down if $|1-c| \ll 1$.

Turning now to the effect of the last term in (8.5), assuming c to be real and equating the imaginary part of the entire equation to zero yields

$$F_i(z) = [(6.4)]\, z^{-3} + r(\alpha c)^{-1} e^{\alpha} \sinh \alpha F_i(Z), \tag{8.9}$$

where $[(6.4)]$ stands for the right-hand side of equation (6.4). We observe that the second term on the right-hand side of (8.9) will be of the same order of magnitude as the left-hand side if $(\alpha c)\, e^{-\alpha}$ cosech α is the of same order of magnitude as r; for example, if $r = 10^{-3}$ the coefficient of $F_i(Z)$ will exceed unity for $\alpha > 4{\cdot}6$. This term will represent an energy sink if $0 < c < 1$, since then $Z < 0$ and $F_i(Z) > 0$. If $c > 1$, $Z > 0$, and $F_i(Z) > (<)\, 0$ corresponds to an energy source (sink). We infer that the effect of the upper fluid cannot be neglected for sufficiently large α and that it will introduce a new class of modes having wave speeds in excess of the interfacial velocity. We then would expect far more complicated neutral curves, as in Lock's (1954) problem.

We again emphasize that the conclusions of the last two paragraphs are not likely to be valid if α_g is small.

9. Conclusions

We conclude that: (*a*) a thin film of liquid having a linear mean-velocity profile and bounded by a fixed wall and a free surface is unstable for sufficiently large values of both the Reynolds number R and the Weber number W; (*b*) a sufficient condition for stability of such a film with respect to small disturbances is either $R < 203$ or $W < 3$; (*c*) such a film constitutes an adequate model for a two-fluid configuration for sufficiently small values of the density ratio r, but the latter restriction may carry with it the requirement that the kinematic viscosity of the upper fluid be *large* compared with that of the film.

We have not established the conditions that determine whether the instability examined here, depending on energy transfer through the Reynolds stress in the wall layer, dominates that which could arise through energy transfer from the upper fluid, either through the Reynolds stress in the layer adjacent to the interface or through the Reynolds stress associated with profile curvature. An analysis comprising all of these models of energy transfer would be extremely involved, but it seems likely that their relative importance could be established by direct observations of wavelength and wave speed, the latter being decisive with respect to the role of Reynolds stress at the wall.

I am indebted to Mr David Giedt for programming and supervising the numerical calculations of §§ 6 and 7 and of $\mathscr{F}(z)$ in the appendix.

Appendix

The computation of $F(z)$

The function $F(z)$, as defined by (4.4) and (3.3), was introduced originally by Tietjens (1925), who computed it for $z = 0\,(0{\cdot}5)\,5$, while Lin (1945; see L 3.6) recomputed it for $z = 1\,(0{\cdot}2)\,5$; see also Holstein (1950). We have repeated the computation once again in order to obtain more accurate values of $z^3F_i(z)$ and also to extend the results to larger values of z.

Following Lin, we find it more convenient to consider

$$\mathscr{F}(z) = [1-F(z)]^{-1} = -z\int_{\infty}^{-z} f_3(\zeta)\,d\zeta \Big/ \int_{\infty}^{-z} \zeta f_3(\zeta)\,d\zeta, \tag{A 1}$$

where f_3 is given by (3.3) and satisfies

$$f''(\zeta) - i\zeta f(\zeta) = 0, \quad f(\infty) = 0. \tag{A 2a, b}$$

Integrating (A 2 *a*) subject to (A 2 *b*), we obtain

$$\int_{\infty}^{\zeta} \zeta f_3(\zeta)\, d\zeta = -i f_3'(\zeta), \tag{A 3}$$

whence (A 1) reduces to

$$\mathscr{F}(z) = \frac{z}{i f_3'(-z)} \int_{\infty}^{-z} f_3(\zeta)\, d\zeta. \tag{A 4}$$

We may reduce the remaining integral to one more suitable for numerical evaluation with the aid of the result [W 13.21 (8), where W indicates Watson's treatise (1945)]

$$\int_0^{\infty} f_3(\zeta)\, d\zeta = \left(\frac{2}{\sqrt{3}}\right) e^{-11i\pi/12}. \tag{A 5}$$

We also find it convenient to introduce

$$g(z) = +\frac{e^{3i\pi/4}}{2\sqrt{3}} f_3(\zeta) = \mathrm{Ai}\,(z\, e^{-5i\pi/6}), \quad (z = \zeta\, e^{i\pi}), \tag{A 6 a, b}$$

where Ai denotes the Airy function of the first kind. Substituting (A 5) and (A 6 *a, b*) in (A 4) and (A 2 *a*), we obtain

$$\mathscr{F}(z) = \frac{z}{i g'(z)} \left[\tfrac{1}{3} e^{-i\pi/6} + \int_0^z g(z)\, dz\right] \tag{A 7}$$

and

$$g''(z) + i z g(z) = 0. \tag{A 8}$$

We also note that

$$g(0) = [3^{\frac{2}{3}}\Gamma(2/3)]^{-1}, \quad g'(0) = [3^{\frac{1}{3}}\Gamma(1/3)]^{-1} e^{i\pi/6}. \tag{A 9 a, b}$$

The differential equation (A 8) was integrated numerically, with a z-interval of 0·01, to obtain $\mathscr{F}_r$, $\mathscr{F}_i$, $\mathscr{F}'_r$, $\mathscr{F}'_i$, F_r and F_i for $z = 0{\cdot}1\,(0{\cdot}1)\,10$ and $-z = 0{\cdot}1\,(0{\cdot}1)\,6$. The results are presented in table 1.

We may obtain the asymptotic series for $\mathscr{F}(z)$ by carrying out the integration in (A 4) according to [W 10.74 (51)]

$$\zeta \int_{\infty}^{\zeta} f(\zeta)\, d\zeta = -i[w S_{0,\frac{1}{3}}(w) f'(\zeta) + w^2 S_{-1,-\frac{2}{3}}(w) f(\zeta)], \tag{A 10}$$

where $S_{\mu,\nu}$ denotes Lommel's function and

$$w = \tfrac{2}{3}\zeta^{\frac{3}{2}} e^{3i\pi/4}. \tag{A 11}$$

Introducing the known, asymptotic series for the Hankel and Lommel functions then yields

$$\mathscr{F}(z) \sim 1 + \frac{e^{i\pi/4}}{z^{\frac{3}{2}}} + \frac{9}{4}\frac{e^{i\pi/2}}{z^3} + \frac{263}{32}\frac{e^{3i\pi/4}}{z^{\frac{9}{2}}} + O(z^{-6}) \tag{A 12 a}$$

$$\sim 1 + \frac{e^{-5i\pi/4}}{\zeta^{\frac{3}{2}}} + \frac{9}{4}\frac{e^{-i\pi/2}}{\zeta^3} + \frac{263}{32}\frac{e^{i\pi/4}}{\zeta^{\frac{9}{2}}} + O(\zeta^{-6}). \tag{A 12 b}$$

We find that (A 12 *a*) is adequate only for very large z, say $z > 10$, and is still 5 % in error at $z = 8$. On the other hand, (A 12 *b*) gives better than 0·1 % accuracy for $\zeta = -z > 5$.

TABLE 1. $\mathscr{F}_r(z)$, $\mathscr{F}_i(z)$, $\mathscr{F}'_r(z)$, $\mathscr{F}'_i(z)$, $F_r(z)$, $F_i(z)$ and $z^3F_i(z)$ versus $z = 0{\cdot}1\,(0{\cdot}1)\,10{\cdot}0$ and $-0{\cdot}1\,(0{\cdot}1)-6{\cdot}0$. The position of the decimal point for the dependent variables is given by the last two digits, preceded by a space; for example,

$$0{\cdot}2901\quad 01 \equiv 2{\cdot}901, \qquad 0{\cdot}9302-02 \equiv 0{\cdot}009302.$$

The decimal point for the independent variable is placed in its natural position.

z	$\mathscr{F}_r$	$\mathscr{F}_i$	$\mathscr{F}'_r$	$\mathscr{F}'_i$	F_r	F_i	z^3F_i
0·1	−0·1183 00	−0·7778−01	−0·1249 01	−0·9281 00	0·6901 01	−0·3880 01	−0·3880−02
0·2	−0·2489 00	−0·1895 00	−0·1355 01	−0·1328 01	0·3544 01	−0·1936 01	−0·1549−01
0·3	−0·3867 00	−0·3485 00	−0·1380 01	−0·1879 01	0·2427 01	−0·1286 01	−0·3472−01
0·4	−0·5188 00	−0·5712 00	−0·1221 01	−0·2605 01	0·1871 01	−0·9593 00	−0·6139−01
0·5	−0·6194 00	−0·8741 00	−0·7156 00	−0·3467 01	0·1540 01	−0·7616 00	−0·9521−01
0·6	−0·6438 00	−0·1263 01	0·3311 00	−0·4283 01	0·1320 01	−0·6284 00	−0·1357 00
0·7	−0·5321 00	−0·1716 01	0·1995 01	−0·4641 01	0·1165 01	−0·5317 00	−0·1824 00
0·8	−0·2345 00	−0·2159 01	0·3949 01	−0·4029 01	0·1050 01	−0·4578 00	−0·2344 00
0·9	0·2409 00	−0·2484 01	0·5403 01	−0·2315 01	0·9613 00	−0·3989 00	−0·2908 00
1·0	0·8061 00	−0·2605 01	0·5689 01	−0·1210 00	0·8916 00	−0·3503 00	−0·3503 00
1·1	0·1342 01	−0·2522 01	0·4895 01	0·1672 01	0·8356 00	−0·3090 00	−0·4113 00
1·2	0·1770 01	−0·2298 01	0·3640 01	0·2662 01	0·7897 00	−0·2731 00	−0·4719 00
1·3	0·2072 01	−0·2012 01	0·2445 01	0·2973 01	0·7516 00	−0·2412 00	−0·5298 00
1·4	0·2268 01	−0·1717 01	0·1521 01	0·2897 01	0·7197 00	−0·2122 00	−0·5821 00
1·5	0·2386 01	−0·1438 01	0·8724 00	0·2659 01	0·6926 00	−0·1853 00	−0·6255 00
1·6	0·2450 01	−0·1186 01	0·4354 00	0·2385 01	0·6693 00	−0·1601 00	−0·6558 00
1·7	0·2478 01	−0·9606 00	0·1412 00	0·2129 01	0·6491 00	−0·1360 00	−0·6683 00
1·8	0·2481 01	−0·7590 00	−0·6314−01	0·1910 01	0·6314 00	−0·1128 00	−0·6576 00
1·9	0·2467 01	−0·5774 00	−0·2138 00	0·1727 01	0·6157 00	−0·8996−01	−0·6171 00
2·0	0·2439 01	−0·4126 00	−0·3339 00	0·1574 01	0·6014 00	−0·6742−01	−0·5393 00
2·1	0·2401 01	−0·2619 00	−0·4378 00	0·1443 01	0·5883 00	−0·4491−01	−0·4159 00
2·2	0·2352 01	−0·1235 00	−0·5334 00	0·1325 01	0·5760 00	−0·2227−01	−0·2371 00
2·3	0·2294 01	0·3379−02	−0·6242 00	0·1213 01	0·5641 00	0·6422−03	0·7814−02
2·4	0·2227 01	0·1191 00	−0·7107 00	0·1101 01	0·5523 00	0·2394−01	0·3310 00
2·5	0·2152 01	0·2234 00	−0·7912 00	0·9834 00	0·5403 00	0·4772−01	0·7456 00
2·6	0·2069 01	0·3155 00	−0·8625 00	0·8584 00	0·5277 00	0·7202−01	0·1266 01
2·7	0·1980 01	0·3948 00	−0·9208 00	0·7254 00	0·5143 00	0·9685−01	0·1906 01
2·8	0·1886 01	0·4604 00	−0·9625 00	0·5855 00	0·4995 00	0·1222 00	0·2682 01
2·9	0·1788 01	0·5118 00	−0·9845 00	0·4418 00	0·4831 00	0·1479 00	0·3608 01
3·0	0·1689 01	0·5487 00	−0·9851 00	0·2981 00	0·4646 00	0·1739 00	0·4695 01
3·1	0·1592 01	0·5715 00	−0·9640 00	0·1593 00	0·4435 00	0·1998 00	0·5952 01
3·2	0·1497 01	0·5809 00	−0·9225 00	0·2965−01	0·4195 00	0·2252 00	0·7379 01
3·3	0·1408 01	0·5779 00	−0·8632 00	−0·8679−01	0·3922 00	0·2495 00	0·8966 01
3·4	0·1325 01	0·5641 00	−0·7895 00	−0·1871 00	0·3611 00	0·2719 00	0·1069 02
3·5	0·1250 01	0·5411 00	−0·7055 00	−0·2695 00	0·3264 00	0·2915 00	0·1250 02
3·6	0·1184 01	0·5108 00	−0·6151 00	−0·3333 00	0·2881 00	0·3071 00	0·1433 02
3·7	0·1127 01	0·4751 00	−0·5223 00	−0·3787 00	0·2468 00	0·3174 00	0·1608 02
3·8	0·1080 01	0·4356 00	−0·4304 00	−0·4068 00	0·2036 00	0·3213 00	0·1763 02
3·9	0·1041 01	0·3942 00	−0·3424 00	−0·4194 00	0·1600 00	0·3180 00	0·1886 02
4·0	0·1011 01	0·3522 00	−0·2603 00	−0·4183 00	0·1180 00	0·3072 00	0·1966 02
4·1	0·9889 00	0·3109 00	−0·1859 00	−0·4058 00	0·7976−01	0·2893 00	0·1994 02
4·2	0·9737 00	0·2714 00	−0·1203 00	−0·3842 00	0·4699−01	0·2656 00	0·1968 02
4·3	0·9646 00	0·2343 00	−0·6392−01	−0·3556 00	0·2103−01	0·2378 00	0·1891 02
4·4	0·9606 00	0·2004 00	−0·1708−01	−0·3222 00	0·2384−02	0·2081 00	0·1773 02
4·5	0·9608 00	0·1700 00	0·2044−01	−0·2859 00	−0·9187−02	0·1785 00	0·1627 02
4·6	0·9644 00	0·1433 00	0·4911−01	−0·2483 00	−0·1455−01	0·1507 00	0·1467 02
4·7	0·9704 00	0·1203 00	0·6962−01	−0·2109 00	−0·1493−01	0·1258 00	0·1306 02
4·8	0·9781 00	0·1010 00	0·8284−01	−0·1749 00	−0·1164−01	0·1045 00	0·1156 02
4·9	0·9867 00	0·8524−01	0·8974−01	−0·1413 00	−0·5938−02	0·8690−01	0·1022 02
5·0	0·9958 00	0·7267−01	0·9134−01	−0·1108 00	0·1128−02	0·7289−01	0·9111 01
5·1	0·1005 01	0·6297−01	0·8867−01	−0·8371−01	0·8728−02	0·6212−01	0·8241 01
5·2	0·1013 01	0·5580−01	0·8272−01	−0·6041−01	0·1625−01	0·5416−01	0·7616 01
5·3	0·1021 01	0·5076−01	0·7444−01	−0·4092−01	0·2329−01	0·4855−01	0·7227 01
5·4	0·1028 01	0·4749−01	0·6466−01	−0·2514−01	0·2958−01	0·4482−01	0·7057 01
5·5	0·1034 01	0·4562−01	0·5411−01	−0·1285−01	0·3498−01	0·4257−01	0·7082 01
5·6	0·1039 01	0·4482−01	0·4341−01	−0·3737−02	0·3942−01	0·4143−01	0·7276 01
5·7	0·1043 01	0·4478−01	0·3305−01	0·2574−02	0·4292−01	0·4109−01	0·7610 01
5·8	0·1046 01	0·4525−01	0·2339−01	0·6497−02	0·4553−01	0·4130−01	0·8059 01
5·9	0·1048 01	0·4601−01	0·1471−01	0·8452−02	0·4731−01	0·4184−01	0·8594 01
6·0	0·1049 01	0·4689−01	0·7167−02	0·8851−02	0·4836−01	0·4255−01	0·9191 01
6·1	0·1049 01	0·4774−01	0·8349−03	0·8080−02	0·4878−01	0·4329−01	0·9826 01
6·2	0·1049 01	0·4848−01	−0·4280−02	0·6487−02	0·4868−01	0·4397−01	0·1048 02

TABLE 1 (*cont.*)

z	$\mathscr{F}_r$	$\mathscr{F}_i$	$\mathscr{F}'_r$	$\mathscr{F}'_i$	F_r	F_i	z^3F_i
6·3	0·1048 01	0·4902−01	−0·8229−02	0·4376−02	0·4815−01	0·4451−01	0·1113 02
6·4	0·1047 01	0·4935−01	−0·1111−01	0·2002−02	0·4729−01	0·4489−01	0·1177 02
6·5	0·1046 01	0·4942−01	−0·1303−01	−0·4284−03	0·4620−01	0·4506−01	0·1238 02
6·6	0·1045 01	0·4926−01	−0·1415−01	−0·2759−02	0·4495−01	0·4503−01	0·1295 02
6·7	0·1043 01	0·4888−01	−0·1459−01	−0·4876−02	0·4360−01	0·4481−01	0·1348 02
6·8	0·1042 01	0·4830−01	−0·1451−01	−0·6706−02	0·4222−01	0·4440−01	0·1396 02
6·9	0·1040 01	0·4755−01	−0·1403−01	−0·8210−02	0·4084−01	0·4384−01	0·1440 02
7·0	0·1039 01	0·4667−01	−0·1328−01	−0·9376−02	0·3951−01	0·4314−01	0·1480 02
7·1	0·1038 01	0·4568−01	−0·1236−01	−0·1021−01	0·3825−01	0·4234−01	0·1515 02
7·2	0·1037 01	0·4463−01	−0·1136−01	−0·1075−01	0·3706−01	0·4146−01	0·1548 02
7·3	0·1035 01	0·4354−01	−0·1034−01	−0·1101−01	0·3597−01	0·4054−01	0·1577 02
7·4	0·1035 01	0·4244−01	−0·9351−02	−0·1105−01	0·3497−01	0·3959−01	0·1604 02
7·5	0·1034 01	0·4134−01	−0·8436−02	−0·1091−01	0·3406−01	0·3863−01	0·1630 02
7·6	0·1033 01	0·4026−01	−0·7615−02	−0·1063−01	0·3324−01	0·3769−01	0·1654 02
7·7	0·1032 01	0·3922−01	−0·6901−02	−0·1024−01	0·3248−01	0·3676−01	0·1678 02
7·8	0·1031 01	0·3822−01	−0·6294−02	−0·9798−02	0·3180−01	0·3587−01	0·1702 02
7·9	0·1031 01	0·3726−01	−0·5793−02	−0·9317−02	0·3117−01	0·3502−01	0·1727 02
8·0	0·1030 01	0·3635−01	−0·5388−02	−0·8826−02	0·3058−01	0·3421−01	0·1751 02
8·1	0·1030 01	0·3549−01	−0·5067−02	−0·8344−02	0·3004−01	0·3343−01	0·1777 02
8·2	0·1029 01	0·3468−01	−0·4818−02	−0·7884−02	0·2952−01	0·3270−01	0·1803 02
8·3	0·1029 01	0·3392−01	−0·4627−02	−0·7455−02	0·2903−01	0·3201−01	0·1830 02
8·4	0·1028 01	0·3319−01	−0·4481−02	−0·7062−02	0·2855−01	0·3135−01	0·1858 02
8·5	0·1028 01	0·3250−01	−0·4368−02	−0·6708−02	0·2810−01	0·3073−01	0·1887 02
8·6	0·1027 01	0·3185−01	−0·4279−02	−0·6391−02	0·2765−01	0·3014−01	0·1917 02
8·7	0·1027 01	0·3122−01	−0·4206−02	−0·6109−02	0·2721−01	0·2957−01	0·1947 02
8·8	0·1027 01	0·3062−01	−0·4140−02	−0·5859−02	0·2678−01	0·2903−01	0·1978 02
8·9	0·1026 01	0·3005−01	−0·4079−02	−0·5638−02	0·2636−01	0·2851−01	0·2010 02
9·0	0·1026 01	0·2950−01	−0·4017−02	−0·5440−02	0·2595−01	0·2801−01	0·2042 02
9·1	0·1025 01	0·2896−01	−0·3953−02	−0·5262−02	0·2554−01	0·2752−01	0·2074 02
9·2	0·1025 01	0·2844−01	−0·3886−02	−0·5100−02	0·2514−01	0·2705−01	0·2107 02
9·3	0·1025 01	0·2794−01	−0·3815−02	−0·4951−02	0·2475−01	0·2659−01	0·2139 02
9·4	0·1024 01	0·2745−01	−0·3741−02	−0·4812−02	0·2436−01	0·2615−01	0·2172 02
9·5	0·1024 01	0·2698−01	−0·3664−02	−0·4681−02	0·2399−01	0·2572−01	0·2205 02
9·6	0·1024 01	0·2652−01	−0·3584−02	−0·4556−02	0·2362−01	0·2530−01	0·2238 02
9·7	0·1023 01	0·2607−01	−0·3503−02	−0·4436−02	0·2326−01	0·2488−01	0·2271 02
9·8	0·1023 01	0·2563−01	−0·3422−02	−0·4319−02	0·2291−01	0·2448−01	0·2304 02
9·9	0·1022 01	0·2520−01	−0·3341−02	−0·4206−02	0·2257−01	0·2409−01	0·2338 02
10·0	0·1022 01	0·2479−01	−0·3261−02	−0·4096−02	0·2223−01	0·2371−01	0·2371 02
−0·1	0·1047 00	0·5379−01	−0·9804 00	−0·4434 00	−0·6555 01	0·3880 01	−0·3880−02
−0·2	0·1964 00	0·9065−01	−0·8557 00	−0·3020 00	−0·3197 01	0·1937 01	−0·1549−01
−0·3	0·2763 00	0·1156 00	−0·7451 00	−0·2019 00	−0·2080 01	0·1288 01	−0·3477−01
−0·4	0·3459 00	0·1320 00	−0·6492 00	−0·1308 00	−0·1523 01	0·9627 00	−0·6162−01
−0·5	0·4066 00	0·1424 00	−0·5667 00	−0·8004−01	−0·1191 01	0·7671 00	−0·9588−01
−0·6	0·4597 00	0·1485 00	−0·4963 00	−0·4377−01	−0·9699 00	0·6363 00	−0·1374 00
−0·7	0·5062 00	0·1515 00	−0·4360 00	−0·1782−01	−0·8130 00	0·5425 00	−0·1861 00
−0·8	0·5472 00	0·1523 00	−0·3846 00	0·7189−03	−0·6961 00	0·4721 00	−0·2417 00
−0·9	0·5834 00	0·1515 00	−0·3404 00	0·1389−01	−0·6058 00	0·4171 00	−0·3040 00
−1·0	0·6155 00	0·1496 00	−0·3025 00	0·2314−01	−0·5341 00	0·3730 00	−0·3730 00
−1·1	0·6441 00	0·1470 00	−0·2698 00	0·2954−01	−0·4758 00	0·3368 00	−0·4483 00
−1·2	0·6696 00	0·1438 00	−0·2414 00	0·3383−01	−0·4276 00	0·3066 00	−0·5298 00
−1·3	0·6925 00	0·1403 00	−0·2168 00	0·3658−01	−0·3872 00	0·2810 00	−0·6174 00
−1·4	0·7131 00	0·1365 00	−0·1954 00	0·3819−01	−0·3528 00	0·2590 00	−0·7108 00
−1·5	0·7316 00	0·1327 00	−0·1766 00	0·3896−01	−0·3233 00	0·2399 00	−0·8098 00
−1·6	0·7484 00	0·1288 00	−0·1601 00	0·3911−01	−0·2977 00	0·2232 00	−0·9144 00
−1·7	0·7637 00	0·1249 00	−0·1455 00	0·3882−01	−0·2753 00	0·2085 00	−0·1024 01
−1·8	0·7776 00	0·1210 00	−0·1327 00	0·3821−01	−0·2556 00	0·1954 00	−0·1139 01
−1·9	0·7903 00	0·1172 00	−0·1212 00	0·3737−01	−0·2381 00	0·1836 00	−0·1260 01
−2·0	0·8019 00	0·1135 00	−0·1111 00	0·3637−01	−0·2225 00	0·1731 00	−0·1385 01
−2·1	0·8125 00	0·1099 00	−0·1020 00	0·3527−01	−0·2086 00	0·1635 00	−0·1515 01
−2·2	0·8223 00	0·1065 00	−0·9390−01	0·3411−01	−0·1960 00	0·1549 00	−0·1649 01
−2·3	0·8314 00	0·1031 00	−0·8662−01	0·3291−01	−0·1846 00	0·1470 00	−0·1788 01
−2·4	0·8397 00	0·9990−01	−0·8007−01	0·3170−01	−0·1743 00	0·1397 00	−0·1931 01
−2·5	0·8474 00	0·9679−01	−0·7415−01	0·3049−01	−0·1649 00	0·1331 00	−0·2079 01
−2·6	0·8545 00	0·9380−01	−0·6881−01	0·2931−01	−0·1563 00	0·1269 00	−0·2231 01
−2·7	0·8612 00	0·9093−01	−0·6396−01	0·2814−01	−0·1484 00	0·1213 00	−0·2387 01
−2·8	0·8673 00	0·8817−01	−0·5956−01	0·2702−01	−0·1412 00	0·1160 00	−0·2547 01
−2·9	0·8731 00	0·8552−01	−0·5555−01	0·2592−01	−0·1345 00	0·1111 00	−0·2710 01

TABLE 1 (*cont.*)

z	$\mathscr{F}_r$	$\mathscr{F}_i$	$\mathscr{F}'_r$	$\mathscr{F}'_i$	F_r	F_i	z^3F_i
−3·0	0·8785 00	0·8298−01	−0·5190−01	0·2487−01	−0·1283 00	0·1066 00	−0·2878 01
−3·1	0·8835 00	0·8055−01	−0·4856−01	0·2385−01	−0·1226 00	0·1023 00	−0·3049 01
−3·2	0·8882 00	0·7821−01	−0·4550−01	0·2288−01	−0·1172 00	0·9838−01	−0·3224 01
−3·3	0·8926 00	0·7597−01	−0·4269−01	0·2195−01	−0·1123 00	0·9467−01	−0·3402 01
−3·4	0·8967 00	0·7382−01	−0·4011−01	0·2106−01	−0·1077 00	0·9118−01	−0·3584 01
−3·5	0·9006 00	0·7176−01	−0·3774−01	0·2021−01	−0·1033 00	0·8791−01	−0·3769 01
−3·6	0·9043 00	0·6978−01	−0·3555−01	0·1939−01	−0·9931−01	0·8483−01	−0·3958 01
−3·7	0·9077 00	0·6788−01	−0·3353−01	0·1862−01	−0·9552−01	0·8192−01	−0·4149 01
−3·8	0·9110 00	0·6605−01	−0·3166−01	0·1788−01	−0·9197−01	0·7917−01	−0·4344 01
−3·9	0·9141 00	0·6430−01	−0·2993−01	0·1717−01	−0·8862−01	0·7658−01	−0·4543 01
−4·0	0·9170 00	0·6262−01	−0·2832−01	0·1650−01	−0·8547−01	0·7412−01	−0·4744 01
−4·1	0·9197 00	0·6100−01	−0·2683−01	0·1586−01	−0·8251−01	0·7179−01	−0·4948 01
−4·2	0·9224 00	0·5944−01	−0·2544−01	0·1525−01	−0·7970−01	0·6958−01	−0·5155 01
−4·3	0·9248 00	0·5795−01	−0·2414−01	0·1468−01	−0·7705−01	0·6748−01	−0·5365 01
−4·4	0·9272 00	0·5651−01	−0·2293−01	0·1413−01	−0·7455−01	0·6549−01	−0·5578 01
−4·5	0·9294 00	0·5512−01	−0·2180−01	0·1361−01	−0·7217−01	0·6359−01	−0·5794 01
−4·6	0·9315 00	0·5378−01	−0·2075−01	0·1312−01	−0·6992−01	0·6177−01	−0·6013 01
−4·7	0·9336 00	0·5249−01	−0·1976−01	0·1266−01	−0·6778−01	0·6004−01	−0·6234 01
−4·8	0·9355 00	0·5125−01	−0·1885−01	0·1224−01	−0·6575−01	0·5838−01	−0·6457 01
−4·9	0·9373 00	0·5005−01	−0·1801−01	0·1184−01	−0·6381−01	0·5680−01	−0·6682 01
−5·0	0·9391 00	0·4888−01	−0·1724−01	0·1146−01	−0·6197−01	0·5528−01	−0·6910 01
−5·1	0·9408 00	0·4775−01	−0·1656−01	0·1110−01	−0·6020−01	0·5381−01	−0·7138 01
−5·2	0·9424 00	0·4666−01	−0·1595−01	0·1075−01	−0·5851−01	0·5241−01	−0·7369 01
−5·3	0·9440 00	0·4560−01	−0·1544−01	0·1037−01	−0·5687−01	0·5106−01	−0·7601 01
−5·4	0·9455 00	0·4459−01	−0·1501−01	0·9932−02	−0·5528−01	0·4977−01	−0·7836 01
−5·5	0·9470 00	0·4362−01	−0·1465−01	0·9402−02	−0·5374−01	0·4854−01	−0·8076 01
−5·6	0·9484 00	0·4271−01	−0·1431−01	0·8725−02	−0·5223−01	0·4739−01	−0·8322 01
−5·7	0·9499 00	0·4188−01	−0·1392−01	0·7854−02	−0·5075−01	0·4633−01	−0·8580 01
−5·8	0·9512 00	0·4115−01	−0·1335−01	0·6758−02	−0·4932−01	0·4539−01	−0·8857 01
−5·9	0·9525 00	0·4054−01	−0·1245−01	0·5443−02	−0·4796−01	0·4460−01	−0·9160 01
−6·0	0·9537 00	0·4007−01	−0·1097−01	0·3999−02	−0·4671−01	0·4397−01	−0·9498 01

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